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CONTENTS

1. Tauberian Theorems on Generalised Lambert's Series. By V. Ganapathy Iyer 73

2. An Extension of Heilbronn's Class-number Theorem. By S. Chowla 88

3. A List of Self-reciprocal Functions. By Brij Mohan Mehrotra 93

4. " On the Product of Parabolic Cylinder Functions". By S.C. Dhar. D.Sc. 105

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TAUBERIAN THEOREMS ON GENERALISED LAMBERT'S SERIES

BY

V. Ganapathy Iyer.

I. *Introduction.*

1. Let $o<\lambda_1<\lambda_2\ldots.<\lambda_n\ldots.,\ \lambda_n\to\infty$ and $\alpha>0$. We consider the series

$$\sum_1^\infty \frac{a_n\, e^{-\lambda_n s}}{1-e^{-\alpha\lambda_n s}}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where $s=\sigma+it$. The series (1) may be called the Generalised Lambert's Series which, when $\lambda_n=n$, $\alpha=1$, $e^{-s}=x$, reduces to the ordinary Lambert's series. It can be easily proved that the series (1) converges, if it converges at all, in a half-plane $\sigma\geqslant\sigma_0>0$[1]. In what follows it is supposed that the half-plane of convergence is $\sigma>0$ and that $s=\xi\to0$ through positive real values. The object of this paper is to prove for the series (1) the analogue of the following theorems for Dirichlets' series:—

Theorem (A)[2].--Let $f(s)=\Sigma a_n\, e^{-\lambda_n s}$ converge for $R(s)>0$ and let $f(s)\to A$ as $s\to o$ through positive real values. Let, further, $a_n=O\left(\dfrac{\lambda_n-\lambda_{n-1}}{\lambda_n}\right)$. Then $\overset{\infty}{\underset{1}{\Sigma}} a_n=A$.

Theorem (B)[3].— Let $a_n\geqslant0$ and $f(s)=\Sigma a_n\, e^{-\lambda_n s}\sim A\, s^{-\alpha}\, L\left(\frac{1}{s}\right)$ as $s\to0$ through positive real values, where $a\geqslant0$

---

[1] It can be proved, by partial summation, that the two series $\Sigma \dfrac{a_n\, e^{-\lambda_n s}}{1-e^{-\mu_n s}}$ and $\Sigma\, a_n\, e-^{\lambda_n s}$ converge at the same time for $a\sigma>o$. Here $\mu_1<\mu_2\cdots$ and $\mu_n\to\infty$, If $\mu_n=a\lambda_n$, this series becomes that considered in (1).

[2] J. E. Littlewood, "The Converse of Abel's Theorem on power series", Proc. L.M.S., (2), 9 (1910), 434—48. The proof given in this paper assumes $\dfrac{\lambda_n}{\lambda_{n-1}}\to 1$ as $n\to\infty$. For a proof without any restriction on λ_n, see K. Ananda Rau, "On the converse of Abel's Theorem", Journal of the L.M.S., Vol. 3, part 3, 200—205.

[3] Hardy and Littlewood, "Tauberian Theorems concerning power series and Dirichlet's series whose co-efficients are positive". Proc. L.M.S., 2 (13), 1913, 174—191.

and $L(u)=(l_1\, u)^{\alpha_1} \ldots\ldots (l_r\, u)^{\alpha_r}$, where $l_r u=$ [log log ...... (r times)] (u) and the first $\alpha_k \neq 0$ is positive. Then $A_n = \sum_1^n a_r \sim \frac{A}{\Gamma(\alpha+1)} \lambda_n^{\alpha} L(\lambda_n)$.

2. In order to prove analogous theorems for the series (1), the following theorem, due to N. Wiener[4], is used in the sequel:—

Theorem (C)[5].—Let $f(\omega)$ be a function bounded in (o, ∞) Let $\omega\, N(\omega)$ be a function belonging to L_1 in $(0, \infty)$ such that

$$h(u)=\frac{1}{\sqrt{2\pi}} \int_0^\infty N(\omega)\, \omega^{iu}\, d\omega$$

does not vanish for real values of u. Let

$$\underset{\xi \to 0}{Lt} \int_0^\infty N(\xi\omega)\, f(\omega)\, d\omega = A \int_0^\infty N(\omega)\, d\omega.$$

Then, if $\omega\, M(\omega)$ be any function belonging to L_1 in $(0, \infty)$, we have,

$$\underset{\xi \to 0}{Lt} \int_0^\infty M(\xi\omega)\, f(\omega)\, d(\omega) = A \int_0^\infty M(\omega)\, d\omega$$

3. The analogue of theorem (A) is proved in section II and that of theorem (B) in section III. Finally some examples to illustrate these theorems are given.

II

4. *Theorem* (*1*).—Let

$$f(\xi)=\alpha\xi \sum_1^\infty \frac{a_n\, e^{-\lambda_n \xi}}{1-e^{-\alpha\lambda_n \xi}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

converge for every $\xi > 0$, and let

$f(\xi) \to A$, as $\xi \to 0$ through positive values.

Let

$$\zeta(s, a) = \frac{1}{a^s} + \frac{1}{(a+1)^s} + \frac{1}{(a+2)^s} + \ldots\ldots\ldots\ldots (3)$$

---

[4] N. Wiener, "Tauberian Theorems", Annals of Mathematics, second series, Vol. 33, No. 1, pp. 1—100, Jan. 1932. See page 25, theorem VIII. Theorem (C) quoted above is obtained from the last theorem by a simple variable transformation.

[5] This theorem can be stated by saying that for a bounded sequence if an average of a particular kind, with some restriction, exists, then averages of all kinds, where the averaging function belongs to L_1, also exist.

have no zeroes on the line $R(s)=1$ when $\alpha=\frac{1}{\alpha}$. And let

$$a_n=O\ (\lambda_n-\lambda_{n-1}) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Then $\sum\limits_{1}^{\infty}\frac{a_n}{\lambda_n}$ converges to the value A. In particular if $a=1$, (3) does not vanish on $R(s)=1$ and so the theorem is true. Conversely, let $\sum\limits_{1}^{\infty}\frac{a_n}{\lambda_n}$ converge to A. Then, without any further assumption, (2) converges for $\xi>0$ and $f(\xi)\rightarrow A$ as $\xi\rightarrow 0$.

Proof.—The proof is based on the following lemmas:—

Lemma (1-a).—Let $h(\xi)=\dfrac{a\,\xi\,e^{-\xi}}{1-e^{-\alpha\,\xi}}-1$. Then, there are positive constants $\eta=\eta(\alpha)$, $\kappa=\kappa(\alpha)$ depending on α only such that

$$|h(\xi)|\leqslant\kappa(\alpha)\xi,$$

for $0\leqslant\xi\leqslant\eta$.

Proof.—Let $\beta>0$ and $0\leqslant\xi\leqslant 1$. Then,

$$h(\xi)-\beta\,\xi=\frac{1}{1-e^{-\alpha\xi}}\left\{\alpha\,\xi\,e^{-\xi}-(1+\beta\,\xi)\ (1-e^{-\alpha\xi})\right\}$$

$$=\frac{\alpha\,\xi^2}{1-e^{-\alpha\,\xi}}\left\{\left(\frac{\alpha}{2}-1-\beta\right)+\xi\,\kappa(\xi,\alpha,\beta)\right\}$$

where $|\kappa(\xi,\alpha,\beta)|\leqslant\alpha\,e+e^{\alpha}(1+\beta)$ for $0\leqslant\xi\leqslant 1$. Hence if $\beta=|\frac{\alpha}{2}-1|+1$, $h(\xi)-\beta\xi\leqslant 0$ for $0\leqslant\xi\leqslant\dfrac{1}{\alpha e+e^{\alpha}(1+\beta)}=\eta(\alpha)$ Similarly by considering $h(\xi)+\beta\xi$, we see that, by taking $\beta=|\frac{\alpha}{2}-1|+1$, $h(\xi)+\beta\xi\geqslant 0$ for $o\leqslant\xi\leqslant\dfrac{1}{\alpha e+e^{\alpha}(1+\beta)}=\eta(\alpha)$.

Hence,

$$|h(\xi)|\leqslant\beta\,\xi=\left(\left|\frac{\alpha}{2}-1\right|+1\right)\xi=\kappa(\alpha)\xi$$

for $0\leqslant\xi\leqslant\eta(\alpha)$. So the lemma is proved.

Lemma (1-b).—Let $A(\omega)=\sum\limits_{\lambda_n\leqslant\omega}\frac{a_n}{\lambda_n}$, $A(\omega)=0$ $0\leqslant\omega\leqslant\lambda_1$. If $f(\xi)=O(1)$ as $\xi\rightarrow 0$ and a_n satisfies the condition (4), then $A(\omega)=O(1)$ as $\omega\rightarrow\infty$.

Proof.—There is a C such that

$$|a_n|\leqslant C(\lambda_n-\lambda_n-1).$$

Now, we have,

$$f(\xi)-A(\omega)=\sum_{\lambda_n\leqslant\omega}\frac{a_n}{\lambda_n}\left\{\frac{\alpha\,\lambda_n\xi\,e^{-\lambda_n\xi}}{1-e^{-\alpha\lambda_n\xi}}-1\right\}+\sum_{\lambda_n>\omega}\frac{a_n\alpha\,\xi\,e^{-\lambda_n\xi}}{1-e^{-\alpha\lambda_n\xi}}$$

$$=S_1+S_2\text{, say}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

By using Lemma (1-a), we see that, if $\omega\ \xi=\eta(\alpha)$,

$$| S_1 | \leqslant \sum_{\lambda n\leqslant\omega} C \left(\frac{\lambda n-\lambda n-1}{\lambda n}\right) \kappa(\alpha)\ \lambda n\xi$$

$$\leqslant C\ \xi\omega\ \kappa(\alpha)=C\ \eta(\alpha)\ \kappa(a) \ldots\ldots\ldots\ldots\ldots \quad (6)$$

Also

$$| S_2 | \leqslant C\ \alpha\ \xi \sum_{\lambda n>\omega} \frac{(\lambda n-\lambda n-1)\ e^{-\lambda_n\xi}}{1-e^{-\alpha\lambda_n\xi}}$$

$$\leqslant C\ \alpha\ \xi \sum_{\lambda n>\omega} \lambda n \left| \triangle\left(\frac{e^{-\lambda_n\xi}}{1-e^{-\lambda_n\xi}}\right)\right|$$

$$\leqslant C\ \alpha\ \xi \sum_{\lambda n>\omega} \int_{\lambda n}^{\lambda n+1} x \left| \frac{d}{dx}\ \frac{e^{-x\xi}}{1-e^{-\alpha x\xi}}\right| dx$$

$$\leqslant C\ \alpha\ \xi \int_{\omega}^{\infty} (x\xi) \left| \frac{d}{d(x\xi)}\left(\frac{e^{-x\xi}}{1-e^{-\alpha x\xi}}\right)\right| dx$$

$$=C\ \alpha \int_{\eta(\alpha)}^{\infty} x \left| \frac{d}{dx}\left(\frac{e^{-x}}{1-e^{-\alpha x}}\right)\right| dx \ldots\ldots\ldots \quad (7)$$

Combining (6) and (7), we have from (5), $| f(\xi)-A(\omega)| \leqslant H(a)$ where $\omega\ \xi=\eta(\alpha)$. Hence letting $\omega\to\infty$ through arbitrary values and $\xi\to o$ through the values $\frac{\eta(\alpha)}{\omega}$ we see that since $f(\xi)=O(1)$ as $\xi\to o$, $A(\omega)=O(1)$ as $\omega\to\infty$. So the lemma is proved.

Proof of theorem (1).—Since $f(\xi)\to A$ as $\xi\to 0$, and a_n satisfies (4), we have by Lemma (1-b) that

$$A(\omega)=O(1) \text{ as } \omega\to\infty.$$

Therefore,

$$f(\xi)=a\ \xi \sum_{1}^{\infty} \frac{a_n}{\lambda n}\ \frac{\lambda n\ e^{-\lambda n\xi}}{1-e^{-\alpha\lambda n\xi}}$$

$$=a\ \xi \int_{0}^{\infty} A(\omega)\ \frac{d}{d\omega}\left(\frac{\omega e^{-\omega\xi}}{e^{-\alpha\omega\xi}-1}\right) d\omega$$

$$=\xi \int_{0}^{\infty} A(\omega)\ N(\omega\xi) d\omega$$

$$\to A \text{ as } \xi\to 0 \ldots\ldots\ldots\ldots\ldots\ldots \quad (8),$$

where, $$N(\omega)=\alpha\ \frac{d}{d\omega}\left(\frac{\omega e^{-\omega}}{e^{-\alpha\omega}-1}\right) \ldots\ldots\ldots\ldots\ldots \quad (9).$$

Here we use theorem C. Obviously $\omega\ N(\omega)$ belongs to L_1.

Also

$$
\begin{aligned}
h(u) &= \frac{a}{\sqrt{2\pi}} \int_0^\infty \frac{d}{d\omega} \left(\frac{\omega e^{-\omega}}{e^{-\alpha\omega}-1} \right) \omega^{iu}\, d\alpha \\
&= \mathop{Lt}_{\lambda \to 0} \frac{a}{\sqrt{2\pi}} \int_0^\infty \frac{d}{d\omega} \left(\frac{\omega e^{-\omega}}{e^{-\alpha\omega}-1} \right) \omega^{iu+\lambda}\, d\omega \\
&= \mathop{Lt}_{\lambda \to 0} \frac{\alpha(\lambda+iu)}{\sqrt{2\pi}} \int_0^\infty \frac{\omega^{iu+\lambda} e^{-\omega}}{1-e^{-\alpha\omega}}\, d\omega \\
&= \mathop{Lt}_{\lambda \to 0} \frac{\alpha(\lambda+iu)}{\sqrt{2\pi}} \int_0^\infty \omega^{iu+\lambda}. \left[e^{-\omega} + e^{-\omega(1+\alpha)} + .\,. \right] d\omega \\
&= \mathop{Lt}_{\lambda \to 0} \frac{(\alpha\lambda+iu)}{\sqrt{2\pi}\, \alpha^{1+\lambda+iu}} \Gamma(1+\lambda+iu)\, \zeta\left(1+\lambda+iu, \frac{1}{\alpha}\right) \\
&= \frac{iu}{\sqrt{2\pi}\, \alpha^{iu}} \Gamma(1+iu)\, \zeta\left(1+iu, \frac{1}{\alpha}\right) \\
&\neq 0
\end{aligned}
$$

by hypothesis. The positive quantity λ is introduced to make term by term integration possible. Also $\int_0^\infty N(\omega)\, d\omega = 1$. Hence all the conditions of theorem (C) are satisfied when $N(u)$ has the value (9). So we can replace $N(u)$ by any function $N(u)$ in (8) where $u\, N(u)$ belongs to L_1.

Let
$$
M(x) = \begin{cases} 1, & 0 \leqslant x \leqslant 1 \\ 0, & x > 1. \end{cases}
$$

Then we get from (8), putting $\xi = \frac{1}{\omega}$, that

$$
\frac{1}{\omega} \int_0^\alpha A(t)\, dt \to A, \text{ as } \omega \to \infty \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(10)}.
$$

But the left side of (10) is the Riesz's mean (type λ_n) of the first order for the series $\sum \frac{a_n}{\lambda_n}$ and (10) says that this mean exists. But $\frac{a_n}{\lambda_n} = O\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right)$ by (4). So by a well-known theorem. $A(\omega) \to A$ as $\omega \to \infty$, that is $\sum_1^\infty \frac{a_n}{\lambda_n} = A$.

Conversely let $\sum_1^\infty \frac{a_n}{\lambda_n} = A$. Then $A(\omega) \to A$ as $\omega \to \infty$.

Therefore (10) is valid and can be written as

$$\xi \int_{o}^{\infty} M(\xi\,\omega)\; A(\omega)\; d\omega \to A \text{ as } \xi \to o, \quad \dots\dots\dots \quad (11)$$

where
$$M(x) = \begin{cases} 1, & 0 \leqslant x \leqslant 1 \\ 0, & x > 1. \end{cases}$$

Also, $\dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{o}^{\infty} M(x)\; x^{iu}\, dx = \dfrac{1}{\sqrt{2\pi}(1+iu)} \neq 0$ for any real u.

So by theorem (C) we can replace $M(x)$ in (11) by $N(u)$ given by (9). So the conclusion stated by the relation (8) is valid which proves the converse part of theorem (1).

5. *Remark on the converse part of theorem* (1).—The converse part of theorem (1) is the usual Abelian theorem for the series (2). It is not necessary to use theorem (C) to prove it. An alternative method is given below:—

We have, $A(\omega) \to A$ by hypothesis. So,

$$f(\xi) - A = \alpha\,\xi \int_{\omega}^{\infty} [A(\omega) - A]\; \frac{d}{d\omega}\left(\frac{\omega e^{-\omega\xi}}{e^{-\alpha\omega\xi} - 1}\right) d\omega$$

$$= \int_{\omega}^{\infty} \left[A\left(\frac{\omega}{\xi}\right) - A\right] N(\omega)\; d\omega$$

$$= \int_{o}^{\sqrt{\xi}} + \int_{\sqrt{\xi}}^{\infty}$$

$$= O(1)\; \sqrt{\xi} + 0(1)\; O(1), \text{ as } \xi \to 0$$

$$= o(1) \text{ as } \xi \to 0$$

because, $A\left(\frac{\omega}{\xi}\right) = O(1)$ in $0 \leqslant \omega \leqslant \sqrt{\xi}$ and $A\left(\frac{\omega}{\xi}\right) = 0\;(1)$ in $\sqrt{\xi} \leqslant \omega < \infty$ as $\xi \to 0$ and $\displaystyle\int_{o}^{\infty} |\,N(u)\,|\; du$ is finite. Hence $f(\xi) \to A$ as $\xi \to 0$ as is to be proved.

III

6. *Theorem* (2).—Let $a_n \geqslant o$ and $A(\omega) = \sum\limits_{\lambda_n \leqslant \omega} \dfrac{a_n}{\lambda_n}$. Let

$$f(\xi) = a\xi \sum_{1}^{\infty} \frac{a_n\, e^{-\lambda_n \xi}}{1 - e^{-\alpha\lambda_n\xi}} \sim A\xi^{-\beta}\; L\left(\frac{1}{\xi}\right) \quad \dots\dots\dots \quad (12)$$

as $\xi \to 0$. Here $\beta \geqslant 0$ and

$$L(u) = (l_1\, u)^{\alpha_1}\; (l_2\, u)^{\alpha_2} \;\dots\; (l_r\, u)^{\alpha_r} \quad \dots\dots\dots\dots \quad (13)$$

where $l_k(u)$=the kth repeated logarithm of u and $(\alpha_1, \dots \alpha_r)$ are *any constants* whatever.[6] Let

$$\zeta\left(1+\beta+iu, \frac{1}{\alpha}\right) \neq 0 \dots\dots\dots\dots\dots\dots\dots \quad (14)$$

for real *u*, which is satisfied when (i) $\alpha=1, \beta\geqslant 0$ and (ii) β greater than some positive constant depending on α when $\alpha\neq 1$. Then,

$$A(\omega) \sim \frac{A\alpha^{\beta}\,\omega^{\beta}\,L(\omega)}{\beta\Gamma(\beta+1)\,\zeta\left(\beta+1, \frac{1}{\alpha}\right)} \dots\dots\dots\dots\dots\dots\dots (15)$$

as $\omega\rightarrow\infty$. Conversely, if (15) be true, then (12) is true without any further assumption on a_n or on the non-vanishing of (14).

Proof.—We shall, first, prove several lemmas:—

Lemma (*2-a*).=Under the hypothesis of theorem (2),

$$A(\omega)=O(\omega^{\beta}\,L(\omega)) \text{ as } \alpha\rightarrow\infty.$$

Proof.—It can be shown by differentiation that $\frac{\alpha x e^{-x}}{1-e^{-\alpha x}}$ is a monotone increasing or decreasing function of x according to the value of α in a certain range $0\leqslant x \leqslant\eta(\alpha)$, where $\eta(\alpha)$ is a positive constant depending on α. So there exists a constant $C(\alpha)$ such that, if $\omega\xi=\eta(\alpha)$, we have,

$$\begin{aligned} 0\leqslant A(\omega) &\leqslant C(\alpha)\, a\xi \sum_{\lambda_n\leqslant\omega} \frac{a_n e^{-\lambda_n\xi}}{1-e^{-\alpha\lambda_n\xi}} \\ &= O\left[f\left(\frac{\eta}{\omega}\right)\right] \\ &= O\left(\omega^{\beta}\,L(\omega)\right) \end{aligned}$$

since $a_n\geqslant 0$. So the lemma is proved.

Lemma (*2-b*)[7].—Let η_r be such that $l_r(\eta_r)=1$, so that $l_k(u)\geqslant 1$, $(k=1, 2, \dots r)$ for $u\geqslant\eta_r$. Let $\overline{L(u)}$ denote the same function as in (13) with $\alpha_1, \dots \alpha_r$ replaced by their moduli. Let

---

[6] Usually it is supposed that in (13) (i) $\alpha_1=\alpha_2= \dots \alpha_r=0$ or (ii) the first $\alpha_k\neq 0$ is positive. But this restriction is not necessary. The same remark holds as regards theorem (B) which could also be proved by the methods of this paper.

[7] The formula (13) does not give real values of $L(u)$ for sufficiently small values of u. Since $L(u)$ is considered only for large values of u we suppose that it is defined to take a continuous set of values lying between two positive bounds for $0\leqslant u\leqslant\eta_r$ so that $L(u)$ as well as $\frac{1}{L(u)}$ is bounded for these values of u.

h be a positive constant and u^o a sufficiently great positive quantity. Then,

$$(i)\quad \frac{L\left(\frac{u}{\xi}\right)}{L\left(\frac{1}{\xi}\right)}=O\left\{\overline{L\left(\frac{1}{\xi}\right)}\right\}\text{ uniformly for } 0<u\leqslant\xi\eta_r.$$

$$(ii)\quad \frac{L\left(\frac{u}{\xi}\right)}{L\left(\frac{1}{\xi}\right)}=O\left\{\overline{L\left(\frac{1}{\xi}\right)}\right\}^2\text{ uniformly for } \xi\eta_r\leqslant u\leqslant\frac{1}{\left(\log\frac{1}{\xi}\right)^h}$$

$$(iii)\quad \frac{L\left(\frac{u}{\xi}\right)}{L\left(\frac{1}{\xi}\right)}\rightarrow 1\text{ uniformly for }\frac{1}{\left(\log\frac{1}{\xi}\right)^h}\leqslant u\leqslant\left(\log\frac{1}{\xi}\right)^h$$

as $\xi\rightarrow o$.

$$(iv)\quad \frac{L\left(\frac{u}{\xi}\right)}{L\left(\frac{1}{\xi}\right)}=O\left\{\overline{L(u)}\right\}\text{ for } u\geqslant u_0\text{ uniformly.}$$

Here the O-signs refer to the variable ξ as it tends to zero.

Proof.—(*i*) This follows from the convention adopted in footnote (7).

(*ii*) We have, $\Theta\leqslant L\left(\frac{u}{\xi}\right)\leqslant\overline{L\left(\frac{1}{\xi\left(\log\frac{1}{\xi}\right)^h}\right)}\leqslant\overline{L\left(\frac{1}{\xi}\right)}$

for $\xi\eta_r\leqslant u\leqslant\frac{1}{\left(\log\frac{1}{\xi}\right)^h}$ and $\frac{1}{L\left(\frac{1}{\xi}\right)}\leqslant\overline{L\left(\frac{1}{\xi}\right)}$. So (*ii*) is true.

(*iii*) We have, in $\frac{1}{\left(\log\frac{1}{\xi}\right)^h}\leqslant u\leqslant\left(\log\frac{1}{\xi}\right)^h,$

$$l_1\left(\frac{u}{\xi}\right)=l_1\left(\frac{1}{\xi}\right)+l_1(u)$$

$$=l_1\left(\frac{1}{\xi}\right)\left\{1+\theta_1\right\},\text{ where } |\,\theta_1\,|\leqslant\frac{hl_2\,\frac{1}{\xi}}{l_1\left(\frac{1}{\xi}\right)},$$

$$l_2\left(\frac{u}{\xi}\right)=l_2\left(\frac{1}{\xi}\right)\left\{1+\theta_2\right\},\text{ where } |\,\theta_2\,|=O\left(\frac{l_2\,\frac{1}{\xi}}{l_1\,\frac{1}{\xi}\;l_2\,\frac{1}{\xi}}\right)$$

and so on. Since $\theta_1, \theta_2\ldots$ tend to zero uniformly as $\xi\rightarrow 0$ we get (*iii*) whatever $\alpha_1, \alpha_2, \ldots \alpha_r$ be.

(*iv*) Let $u_0 > \eta_r$. We have

$l_1\left(\frac{u}{\xi}\right) = l_1(u) + l_1\left(\frac{1}{\xi}\right)$, so that

$$1 \leqslant \frac{l_1\left(\frac{u}{\xi}\right)}{l_1\left(\frac{1}{\xi}\right)} \leqslant 2\, l_1(u), \text{ if } \xi \text{ is sufficiently small.}$$

Hence,

$$1 \leqslant \frac{l_2\left(\frac{u}{\xi}\right)}{l_2\left(\frac{1}{\xi}\right)} \leqslant l_2(u) + \log\, 2 \text{ and so on.}$$

Therefore in all cases,

$$\frac{L\left(\frac{u}{\xi}\right)}{L\left(\frac{1}{\xi}\right)} = O\left[L(u)\right]$$

uniformly for all $u \geqslant u_0$. So the lemma is proved.

Lemma (2-*c*).—Let $g(\xi) = \xi \int_0^\infty \left| \frac{L(x)}{L\left(\frac{1}{\xi}\right)} - 1 \right| H(\xi x)\, dx$

where $H(u)$ is

(*i*) bounded over any finite range

(*ii*) $\Theta\,(e^{-u\lambda})$, $\lambda > 0$ as $u \to \infty$.

Then $g(\xi) = 0(1)$ as $\xi \to 0$. In particular the lemma is true if $H(u) = u^\beta \left| \frac{d}{du}\left(\frac{u\, e^{-u}}{e^{-\alpha u} - 1} \right) \right|$ where $\beta \geqslant 0$.

Proof.—Let $h = 1 + 2\,|\alpha_1|$ in Lemma (2-*b*). We have,

$$g(\xi) = \int_0^\infty \left| \frac{L\left(\frac{x}{\xi}\right)}{L\left(\frac{1}{\xi}\right)} - 1 \right| H(x)\, dx$$

$$= \int_0^{\xi\eta_r} + \int_{\xi\eta_r}^{\left(\log\frac{1}{\xi}\right)^{-h}} + \int_{\left(\log\frac{1}{\xi}\right)^{-h}}^{u_0} + \int_{u_0}^{\infty}$$

$$= S_1 + S_2 + S_3 + S_4, \text{ say.}$$

By Lemma (2-*b*) (*iv*) and the second condition imposed on $H(u)$, we can find u_0 so that $|S_4| < \frac{\epsilon}{2}$. Let u_0 be so chosen and fixed. Now, by Lemma (2-*b*) (*i*), (*ii*) and (*iii*) we have if ξ is sufficiently small.

$$S1=O\left[\xi\, L\left(\frac{1}{\xi}\right)\right].$$

$$S2=O\left[\frac{\left\{L\left(\frac{1}{\xi}\right)\right\}^{2}}{\left(\log\frac{1}{\xi}\right)^{1+2\,|\alpha_1|}}\right]$$

$$S3=\underset{\left(\log\frac{1}{\xi}\right)^{-h}\leqslant x\leqslant u_0}{\mathrm{Max}:}\left|\frac{L\left(\frac{x}{\xi}\right)}{L\left(\frac{1}{\xi}\right)}-1\right|.\quad O(u_0)$$

$$=0(1)\ O(u_0).$$

Hence we can find a $\xi 0$ such that for $0<\xi\leqslant\xi 0$.

$$S_1+S_2+S_3<\epsilon/2.$$

So $0\leqslant g(\xi)<\epsilon$ for $0<\xi\leqslant\xi_0$. So the lemma is proved.

Lemma (*2-d*).—$\lambda(\omega)=\dfrac{1}{\omega}\displaystyle\int_{0}^{\omega}\left|\frac{L(x)}{L(\omega)}-1\right|d\omega\rightarrow 0$ as $\omega\rightarrow\infty$.

Proof.— In lemma (2-*c*), take $H(x)=1$, $0\leqslant x\leqslant 1$ and $H(x)=0$ for $x>1$ and put $=\xi\ \dfrac{1}{\omega}$.

Lemma (*2-e*)[8]—If $f(x)$ be such that $x\,f'(x)$ increases as x increases to infinity and $\phi(x)$ be a positive increasing function such that $\dfrac{\phi(x)}{x\phi'(x)}\rightarrow 1$, then $f(x)\sim\phi(x)$ involves $f'(x)\sim\phi'(x)$. In particular we can take $\phi(x)=A\omega^{\beta+1}\,L(\omega)$, $\beta\geqslant 0$.

Proof.—For a proof, reference is made to the memoir cited in footnote (8).

Proof of theorem (2).—By Lemma (2-*a*), $A(\omega)=O(\omega^{\beta}\ L(\omega)|$. Let $B(\omega)=\dfrac{A(\omega)}{\omega^{\beta}\ L(\omega)}$, so that $B(\omega)$ is bounded. Then, the relation (12) could be written as

$$\underset{\xi\rightarrow 0}{Lt}\ \frac{\alpha\,\xi}{L\left(\frac{1}{\xi}\right)}\int_{0}^{\infty}B(\omega)\ L(\omega)\ (\omega\xi)^{\beta}\ \frac{d}{d\omega}\left(\frac{\omega e^{-\omega\xi}}{e^{-\alpha\omega\xi}-1}\right)=A\ldots\ldots(16)$$

Since $B(\omega)$ is bounded, we see by using Lemma (2-*c*) that

$$\underset{\xi\rightarrow 0}{Lt}\ \xi\int_{0}^{\infty}B(\omega)\ N(\omega\xi)=A\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17)$$

where $N(\omega)=a\omega^{\beta}\dfrac{d}{d\,\omega}\left(\dfrac{\omega\,e^{-\omega}}{e^{-\alpha\omega}-1}\right)$ ……………………(18)

---

[8] See the paper quoted in footnote (3).

Now we apply theorem (c) to (17). Here,

$$\int_0^\infty N(\omega)\ \omega^{iu} d\omega = \alpha \int_0^\infty \omega^{iu+\beta} \frac{d}{d\omega}\left(\frac{\omega\, e^{-\omega}}{e^{-\alpha\omega}-1}\right) d\omega$$

$$\alpha(\beta+iu)\int_0^\infty \omega^{\beta+iu} \frac{e^{-\omega}}{1-e^{-\alpha\omega}}\, d\omega$$

$$= \frac{(\beta+iu)\ \Gamma(1+\beta+iu}{\alpha^{\beta+iu}}\ \zeta(1+\beta+iu, \frac{1}{\alpha})$$

$$\neq 0$$

by hypothesis for any real u. Now (17) could be written as

$$\xi \int_0^\infty B(\omega)\ N(\omega\xi)\ d\omega \to \frac{A\alpha^\beta}{\beta\,\Gamma(\beta+1)\ \zeta\left(\beta+1, \frac{1}{\alpha}\right)} \int_0^\infty N(\omega)\ d\omega \quad \ldots.(19)$$

as $\xi \to 0$. Since $B(\omega)$ is bounded. we get by theorem (c), by taking

$$M(x) = \begin{cases} x^\beta, & 0 \leqslant x \leqslant 1 \\ 0, & x > 1 \end{cases}$$

that, $$\xi \int_0^{\frac{1}{\xi}} B(\omega)\ \ (\omega\xi)^\beta\ d\omega \to \frac{A\alpha^\beta}{\beta(\beta+1)\ \Gamma(\beta+1)\ \zeta\left(\beta+1, \frac{1}{\alpha}\right)}$$

$$= \frac{K}{\beta+1}, \text{ say} \ldots\ldots\ldots\ldots(20)$$

Putting $\xi = \dfrac{1}{\omega}$, we get

$$\frac{1}{\omega^{\beta+1}} \int_0^\omega \frac{A(t)}{L(t)}\ dt \to \frac{K}{\beta+1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(20')$$

since $A(t) = B(t)\ t^\beta\ L(t)$.

Now, $$\left|\frac{1}{\omega^{\beta+1}\ L(\omega)} \int_0^\omega A(t)\ dt - \frac{1}{\omega^{\beta+1}} \int_0^\omega \frac{A(t)}{L(t)}\ dt\right|$$

$$\leqslant \frac{1}{\omega^{\beta+1}\ L(\omega)} \left| \int_0^\omega A(t) \left| \frac{L(\omega)}{L(t)} - 1 \right| dt \right.$$

$$\leqslant \frac{C}{\omega} \int_0^\omega \left| \frac{L(t)}{L(\omega)} - 1 \right| dt, \text{ since } |A(t)| \leqslant c\, t^\beta\ L(t)$$

$= 0\ (1)$, as $\omega \to \infty$, by Lemma (2-d).

So, $$\int_0^\omega A(t)\ dt \sim \frac{K}{\beta+1}\ \omega^{\beta+1}\ L(\omega) \ldots\ldots\ldots\ldots\ldots\ldots(21)$$

Since $A(t)$ is increasing function of (t), we see that $\omega A(\omega)=\omega . \frac{d}{d\omega} \int_{0}^{\omega} A(t)\, dt$ is increasing. Hence by Lemma (2-*e*), we get

$$A(\omega) \sim K \omega^{\beta} L(\omega) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15)$$

as is to be proved.

Conversely if (15) holds, so does (21) and by Lemma (2-*d*), (20′) and hence (20). The relation (20) could be written as

$$\xi \int_{0}^{\infty} B(\omega)\, M(\xi \omega)\, d\omega \rightarrow K \int_{0}^{\infty} M(\omega)\, d\omega$$

and $\int_{0}^{\infty} M(\omega)\, \omega^{iu}\, d\omega = \frac{1}{1+\beta+iu} \neq 0$ for any real u.

Hence we can replace $M(\omega)$ by $N(\omega)$ given by (18) and conclude that (17) is valid. By using Lemma (2-*c*) we see that (16) holds which is nothing but the relation (12).

7. *Remark.*—By using the Lemmas (2-*c*) and (2-*d*), the converse part of theorem (2) could be proved, as in § 5, without the use of theorem (*c*). It is to be noted that the hypothesis $A(\omega) \sim C\omega^{\beta} L(\omega)$ is not equivalent to $A(\lambda_n) \sim C\lambda_n{}^{\beta} L(\lambda_n)$ unless some more restriction is placed on the sequence (λ_n). The relation $A(\omega) \sim C\omega^{\beta} L(\omega)$ implicitly contains the assumption that

(*i*) in case $\beta > 0$, $\frac{\lambda_n}{\lambda_{n-1}} \rightarrow 1$ as $n \rightarrow \infty$,

(*ii*) in case $\beta = 0$, $\alpha_1 = \ldots = \alpha_{\kappa-1} = 0$, $\alpha_{\kappa} \neq 0$, $\frac{l_{\kappa}\, \lambda_n}{l_{\kappa}\, \lambda_{n-1}} \rightarrow 1$ as $n \rightarrow \infty$.

Since. in the direct part of theorem (2), the relation (15) is proved to be a consequence of (12) without any hypothesis on λ_n, it follows that (12) involves—whenever $a_n \geqslant 0$ and (14) is valid—the relation stated in (*i*) or (*ii*) above.

8. *Examples.*—(*i*) The familiar relations

$$\text{(a)} \quad \sum_{1}^{\infty} \frac{\mu(n)}{n} = 0. \qquad \text{(b)} \quad \sum_{1}^{\infty} \frac{\lambda(n)}{n} = 0 \quad \ldots\ldots\ldots\ldots\ldots (22)$$

can be obtained by applying theorem (1) to the known results

$$\text{(a}'\text{)} \quad \xi \sum_{1}^{\infty} \frac{\mu(n)\, e^{-n\xi}}{1-e^{-n\xi}} = \xi e^{-\xi}, \qquad \text{(b}'\text{)} \quad \xi \sum_{1}^{\infty} \frac{\lambda(n)\, e^{-n\xi}}{1-e^{-n\xi}}$$

$$= \xi \sum_{1}^{\infty} e^{-n^2 \xi}. \quad \ldots\ldots\ldots (22')$$

where $\mu(n)$ is the Möbius function and $\lambda(n)=(-1)^\rho$, ρ being the number of prime factors of n counted according to multiplicity and $\lambda(1)=1$.

(*ii*) Let $\phi(n)$ be the number of numbers less than n and prime to it. From $\sum\limits_{d/n} \phi(d)=n$, we get

$$\xi \sum_1^\infty \frac{\phi(n)\, e^{-n\xi}}{1-e^{-n\xi}} = \xi\, e^{-\xi}\, (1-e^{-\xi})^{-2} \sim \frac{1}{\xi} \text{ as } \xi \to 0 \quad ..(23)$$

Since $\phi(n) \geqslant 0$, we get, by applying theorem (2) with $\lambda_n = n$. $\beta=\alpha=1$, that

$$A(\omega)=\sum_{n\leqslant\omega} \frac{\phi(n)}{n} \sim \frac{\omega}{\zeta(2)} = \frac{6\omega}{\pi^2} \quad \ldots\ldots\ldots\ldots (23')$$

From (23') we get

$$\begin{aligned} B(\omega) &= \sum_{n\leqslant\omega} \phi(n) \\ &= \sum_{n\leqslant\omega} \frac{\phi(n)}{n}\, n \\ &= \frac{6\omega^2}{\pi^2}\left\{ 1+0(1) \right\} - \sum_{n\leqslant(\omega-1)} \frac{6n}{\pi^2}\left\{ 1+0(1) \right\} \\ &= \frac{3\omega^2}{\pi^2}\left\{ 1+0(1) \right\} \\ &\sim \frac{3\omega^2}{\pi^2} \quad \ldots\ldots\ldots\ldots (23'') \end{aligned}$$

(*iii*) Let $p_1, p_2, p_3 \ldots\ldots$ be the successive primes, Let $\Pi(x)$ denote the number of primes not exceeding x. Let $c(n)$ denote a positive function of n and let $\tau(n)$ be defined by the relation

$$\tau(n) = \sum_{p_d/n} c(d) \quad \ldots\ldots\ldots\ldots (24).$$

Then the following results are known :—

$$\xi \sum_1^\infty \frac{c(n)\, e^{-p_n\xi}}{1-e^{-p_n\xi}} = \xi \sum_1^\infty \tau(n)\, e^{-n\xi} \quad \ldots\ldots\ldots\ldots (24')$$

$$\left.\begin{aligned} &(\alpha) \quad \Pi(x) \sim \frac{x}{\log x} \\ &(\beta) \quad\quad n \sim \frac{p_n}{\log p_n} \\ &(\gamma) \quad\quad p_n \sim n \log n \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (24'')$$

(*iii-a*) Let $c(n)=1$. Then $\tau(n)$=the number of distinct prime factors of n. Now,

$$A(\omega)=\sum_{p_n\leqslant\omega}\frac{1}{p_n}\sim\log\log\omega\ldots\ldots\ldots\ldots\ldots\ldots(25)$$

by (β) and (γ) of $(24'')$. So by the converse part of theorem (2), we get by $(24')$ that

$$\xi\sum_{1}^{\infty}\tau(n)\ e^{-n\xi}\sim\log\log\frac{1}{\xi}\ .$$

or
$$\sum_{1}^{\infty}\tau(n)\ e^{-n\xi}\sim\xi^{-1}\log\log\frac{1}{\xi}\ldots\ldots\ldots\ldots\ldots\ldots(25')$$

Hence by theorem (B), since $\tau(n)\geqslant0$, we get

$$B(\omega)=\sum_{n\leqslant\omega}\tau(n)\sim\omega\log\log\omega\ldots\ldots\ldots\ldots\ldots\ldots(25'')$$

(*iii-b*) Let $c(n)=n$. Then $\tau(n)$=the sum of the ranks of the prime divisors of n. Here

$$A(\omega)=\sum_{p_n\leqslant\omega}\frac{n}{p_n}\sim\frac{\omega}{(\log\omega)^2}\ldots\ldots\ldots\ldots\ldots\ldots(26)$$

by (β) and (γ) of $(24'')$. Hence, as before,

$$\sum_{1}^{\infty}\tau(n)\ e^{-n\xi}\sim\zeta(2)\ \xi^{-2}\ \frac{1}{\left(\log\frac{1}{\xi}\right)^2}\ \ldots\ldots\ldots\ldots\ldots\ldots(26')$$

So by theorem (B) (See footnote 6),

$$B(\omega)=\sum_{n\leqslant\omega}\tau(n)\sim\frac{\pi^2}{12}\ \frac{\omega^2}{(\log\omega)^2}\ \ldots\ldots\ldots\ldots\ldots\ldots(26'')$$

since $\zeta(2)=\sum\frac{1}{n^2}=\frac{\pi^2}{6}$.

(*iii-c*) Let $c(n)=p_n$. Then $\tau(n)$=the sum of the distinct prime divisors of n. We have,

$$A(\omega)=\sum_{p_n\leqslant\omega}\frac{p_n}{p_n}=\sum_{p_n\leqslant\omega}1\sim\frac{\omega}{\log\omega}\ldots\ldots\ldots\ldots\ldots\ldots(27)$$

by (α), $(24'')$. So we have by $(24')$

$$\sum_{1}^{\infty}\tau(n)\ e^{-n\xi}\sim\zeta(2)\ \xi^{-2}\left(\log\frac{1}{\xi}\right)^{-1}\ \ldots\ldots\ldots\ldots\ldots\ldots(27')$$

Hence by theorem (B),

$$B(\omega)=\sum_{n\leqslant\omega}\tau(n)\sim\frac{\pi^2}{12}\frac{\omega^2}{\log\omega}\ldots\ldots\ldots\ldots\ldots\ldots(27'')$$

9. *Note.* The conclusion of theorem (1) still remains true when the condition (4) on a_n is replaced by the following more general one:—

$$\left.\begin{aligned} &\sum_{\nu=1}^{n} \left|\frac{a_\nu}{\lambda_\nu}\right|^p \left(\frac{\lambda_\nu}{\lambda_\nu-\lambda_{\nu-1}}\right)^p (\lambda_\nu-\lambda_{\nu-1})=O(\lambda_n) \\ \text{that is, } &\sum_{\nu=1}^{n} \left|\frac{a_\nu}{\lambda_\nu-\lambda_{\nu-1}}\right|^p (\lambda_\nu-\lambda_{\nu-1})=O(\lambda_n) \end{aligned}\right\} \ldots (X_p)$$

where p is a positive quantity *greater than unity.* It is easy to prove that if (4) holds, then (X_p) is true for all $p \geqslant 1$ but not conversely. Hence (X_p) is a more general condition than (4). Theorem (1) is valid even when $p=1$, provided in that case the O-sign in X_p $(\equiv X_1)$ be replaced by 0-sign.

AN EXTENSION OF HEILBRONN'S CLASS-NUMBER THEOREM

BY

S. CHOWLA.

Let $h(d)$ denote the number of primitive classes of binary quadratic forms of negative discriminant d. Heilbronn[1] has recently proved that

Theorem I.

$$h(d) \to \infty$$
$$as \quad -d \to \infty.$$

By a slight modification of Heilbronn's argument I show that

Theorem II.

$$\frac{h(d)}{2^t} \to \infty$$
$$as \quad -d \to \infty,$$

where t *is the number of different prime factors of* d.

Both these results were conjectured by Gauss.[2]

Theorem II is equivalent to

Theorem III.

$$p(d) \to \infty$$
$$as \quad -d \to \infty,$$

where p(d) *is the number of primitive classes in the principal genus.*

We shall write

$$h(d) = H, \; p(d) = P.$$

We assume, with Heilbronn, that there is[3] an $m > 0$ and a character $\chi(n)$ (mod m) such that

$$L_0(s) = L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)\, n^{-s}$$

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[1] *Quart, J. of Math.*, (Oxford), 5 (1934), 150—160. Hereafter referred to as Heilbronn. I am very much indebted to Dr. Heilbronn for an advance copy of his manuscript.

[2] *Disquisitiones Arithmeticae* (1801), Art. 303.

[3] If there is no such m, then, by a theorem of Hecke,

$$h(d) > c \frac{\sqrt{|d|}}{\log |d|} \quad (d < -1),$$

so that Theorem II is then certainly true.

vanishes for $s=\rho$ where

$$(1) \qquad \rho=\theta+i\phi \ (\tfrac{1}{2}<\theta<1).$$

We further suppose, as we obviously may, that

$$m\equiv O\,(2)$$

so that

$$(2) \qquad \chi(n)=O \ \text{ for } \ n\equiv O\,(2).$$

We follow the notation of Heilbronn except that the constants implied in our O symbols are independent of H and t.

Lemma I. *If*

$$(3) \qquad a \mid (x^2-d),\ (a, 2d)=1 \qquad [a>1]$$

and if

$$(4) \quad a_s x^2 + b_s xy + c_s y^2 \ (1\leqslant s\leqslant P)$$

are the P *classes in the principal genus then there is an* s *with* 1⩽s⩽P *such that*

$$(5) \quad a^{2P}=a_s x^2+b_s xy+c_s y^2 \ (y\neq O).$$

Proof. Now a^{2P} can be represented by the P classes of the principal genus and not by any of the other $H-P$ classes. The number of representations of a^{2P} by these P forms, is, by a well known theorem,[1] exactly

$$2\,(2P+1)=4\,P+2.$$

Now $a_s x^2+b_s xy+c_s y^2$ can represent a^{2P} with $y=O$ in at most 2 ways. Hence the P classes (4) can represent a^{2P} with $y=O$ in at most $2P$ ways. It follows that a^{2P} must have at least one representation by

$$a_s x^2 + b_s xy + c_s y^2$$

with $y\neq O$ for some s in $1\leqslant s\leqslant P$.

Lemma II. *If*

$$(6) \qquad (a, 2d)=1,\ a \mid (x^2-d) \qquad [a>1]$$

then

$$(7) \qquad a^{2P} \geqslant \sqrt{\frac{3|d|}{16}}$$

Proof. From Lemma I there is an s $(1\leqslant s\leqslant P)$ such that

$$a^{2P}=a_s x^2+b_s xy+c_s y^2 \ (y\neq O)$$

or

$$(8) \quad 4a_s\cdot a^{2P}=\left(2a_s x+b_s y\right)^2-dy^2 \ (y\neq O).$$

---

[1] *See*, for example, Landau, *Varlesungen uber Zahlentheoric*, satz. 204.

Further

$$(9) \qquad 1 \leqslant a_s \leqslant \sqrt{\frac{|d|}{3}}\,.$$

From (8) and (9) we obtain (7).

In what follows a runs through the minima of the H forms of discriminant d.

Lemma III.[2]

(10) *If* $a|d^k$ *for some* $k>0$, *then* $\mu(a) \neq 0$, $a|d$.

Lemma IV. *If*

$$(11) \qquad a \dagger d, \; a \equiv 1 \ (2)$$

then

$$(12) \qquad a^{2P} \geqslant \sqrt{\frac{3|d|}{16}}.$$

Proof. From (11) and (10), $a \dagger d^k$ for any $k>0$. Hence $(a_1, d)=1$ for some a_1 with $a_1>1$ such that $a_1|a$. These give $(a_1, 2d) = 1$.

Further since $a_1 \left| (x^2-d)\right.$, we obtain from Lemma II,

$$(13) \qquad a_1^{2P} \geqslant \sqrt{\frac{3|d|}{16}}\,.$$

(12) follows from (13) since $|a| \geqslant |a_1|$.

Lemma V.[3] *If*

$$A>0, \; A|d, \; \mu(A) \neq 0$$

then there is at most one form of discriminant d *with minimum* A.

Lemma VI.[4] *If under the assumptions of Lemma* V,

$$A \leqslant \left|\sqrt{\frac{d}{4}}\right|$$

then there is at least one form of discriminant d *with minimum* A.

Lemma VII. *For* $\sigma > \frac{1}{2}$,

$$(14) \qquad \left|\sum_a \chi(a)\, a^{-s}\right| \geqslant \frac{1}{4H^2} + O\left(H|d|^{-\frac{\sigma}{2}}\right) + O\left(\frac{H}{|d|^{\sigma/4P}}\right)$$

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[2] Lemma XI of Heilbronn.

[3]. Lemma XII of Heilbronn.

[4]. Lemma XIII of Heilbronn.

Proof. By (2) and (10),

$$\Big| \sum_{a} \chi(a) a^{-s} - \sum_{n \mid d} \chi(n)\, \mu^2(n) n^{-s} \Big|$$

$$= \Big| \sum_{a \nmid d} \chi(a) a^{-s} - \sum_{\substack{n \mid d \\ n \neq a}} \chi(n)\, \mu^2(n) n^{-s} \Big|$$

$$(15) \quad = \Big| \sum_{\substack{a \nmid d \\ a \equiv 1 (2)}} \chi(a) a^{-s} - \sum_{\substack{n \mid d \\ n \neq a}} \chi(n)\, \mu^2(n) n^{-s} \Big|$$

Now by (11) and (12),

$$(16) \quad \sum_{\substack{a \\ a \nmid d \\ a \equiv 1(2)}} \chi(a) a^{-s} = O\left(\frac{H}{|d|^{\sigma/4P}} \right).$$

By Lemmas V and VI,

$$\sum_{\substack{n \mid d \\ n \neq a}} \chi(n) \mu^2(n)\, n^{-s} = O\left(2^t\, |d|^{-\frac{\sigma}{2}} \right)$$

$$(17) \qquad\qquad = O\left(H\, |d|^{-\frac{\sigma}{2}} \right)$$

Further, as in the proof of Lemma X of Heilbronn,

$$(18) \quad \Big| \sum_{n \mid d} \chi(n)\, \mu^2(n) n^{-s} \Big| \geqslant \frac{1}{4H^2}.$$

(14) follows from (15), (16), (17), (18).

Lemma VIII. *For* $\sigma_m < \sigma < 2, s \neq 1,$

$$(19) \quad L_o(s)\, L_2(s) = \zeta(2s) \prod_{p \mid m} \left(1 - p^{-2s}\right) \sum_{a} \chi(a) a^{-s}$$

$$+ O\left\{ \left(|s| + \frac{1}{|s-1|} \right) \left(H\, |d|^{\frac{1}{4} - \frac{\sigma}{2}} + H\, |d|^{-\frac{\sigma}{2}} \right) \right\}.$$

Proof. When $\sigma_m < \sigma < 2, s \neq 1$ we have for $\phi(s)$, the expression on the top of page 156 of Heilbronn,

$$(20) \quad \phi(s) = O\left\{ \left(|s| + \frac{1}{|s-1|} \right) \left(|a^{\sigma}| \,.\, |d|^{-\sigma} + |a^{\sigma-1}| \,.\, |d|^{\frac{1}{2}-\sigma} \right) \right\}$$

Now

$$(21) \quad |a^{\sigma}\, d^{-\sigma}| = \left| \frac{a^2}{d} \right|^{\frac{\sigma}{2}} . \, |d|^{-\frac{\sigma}{2}},$$

$$(22)\qquad \left| a^{\sigma-1} d^{\frac{1}{2}-\sigma} \right| = a^{-\frac{1}{2}} \left| \frac{a^2}{d} \right|^{\frac{\sigma}{2}-\frac{1}{4}} \cdot |d|^{\frac{1}{4}-\frac{\sigma}{2}},$$

$$(23)\qquad a \geqslant 1,\ 3a^2 \leqslant |d|.$$

(19) follows from (20), (21), (22), (23) and the proof of Lemma X in Heilbronn.

Proof of the main result.

We put $s=\rho$ in Lemma VIII. Then we get

$$(24)\qquad 0=\zeta(2\rho) \prod_{p \mid m} \left(1-p^{-2\rho}\right) \sum_{a} \chi(a)\, a^{-\rho} + O\left(H|d|^{\frac{1}{4}-\frac{\theta}{2}} + H|d|^{-\frac{\theta}{2}}\right)$$

From Lemma VII,

$$(25)\qquad \left| \sum_{a} \chi(a) a^{-\rho} \right| \geqslant \frac{1}{4H^2} + O\left(H|d|^{-\frac{\theta}{2}}\right) + O\left(\mathrm{H}|d|^{-\theta/4P}\right).$$

Now

$$\zeta(2\rho) \prod_{p \mid m} \left(1-p^{-2\rho}\right)$$

is absolutely greater than a positive constant independent of d. Hence unless we assume

$$(26)\qquad P \to \infty \text{ as } -d \to \infty,$$

(24) and (25) contradict each other for $-d \to \infty$. Hence (26) is true. This proves our result for negative discriminants d where d or $\frac{d}{4}$ is quadratfrei. The result for general $d<0$ follows now from Lemma I of Heilbronn.

We could also have proved our Theorem II by using the following theorem due to Pepin[1]:

If $(a, 2d)=1,\ a \mid (x^2-d)\qquad [a>1]$

then $a^{2p}=x^2-dy^2\ (y \neq 0)$

where p is the number of classes in the principal genus of reduced primitive non-equivalent binary quadratic forms of negative determinant d.

This result is quoted by Dickson in his History of the theory of numbers, Vol. 3, page 58, without, however, the necessary restriction that a should be prime to d.

July 25, 1934,
WALTAIR.

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[1] *Atti Accad Pont Nuovi Lincei,* 33, 1879—1880, 50—59. This paper has been inaccessible to me.

A LIST OF SELF-RECIPROCAL FUNCTIONS*

BY

BRIJ MOHAN MEHROTRA.

1. The equations

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(y)\cos xy\,dy,$$

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(y)\sin xy\,dy,$$

or, more generally,

$$f(x)=\int_0^\infty \sqrt{xy}\,J_\nu(xy)\,f(y)\,dy, \tag{1.1}$$

where $J_\nu(x)$ is a Bessel function of order $\nu \geqslant -\frac{1}{2}$, have recently engaged the attention of several mathematicians. A function satisfying (1.1) is said to be self-reciprocal for J_ν transforms while a function satisfying (1.1) with the sign of one side changed, is said to be skew-reciprocal for J_ν transforms.

Several solutions of these equations have been given, a few of which might be mentioned: one was given by Bailey† in 1930; two others were given, during the same year, by Hardy and Titchmarsh.‡ Two more solutions were given by Prof. Titchmarsh in an unpublished manuscript a little later.§

Self-reciprocal functions have been appearing from time to time in different papers by different writers.‡‡ A few such functions have recently been discovered by the present writer.|| The object of this note is two-fold: (1) to collect the more

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* This note formed the last chapter of my Ph. D. Thesis submitted to the University of Liverpool in Oct., 1933.

† Bailey (1).

‡ Hardy and Titchmarsh (6).

§ For a brief account of these solutions, *see* Mehrotra (10). A detailed discussion thereof is shortly to appear in the press.

‡‡ A history of these functions has recently been sent for publication by me.

|| These were suggested to me by the work of Glaisher (3).

important of all such functions under one head, (2) to show that from the formal point of view, at least, all these functions fall under one or the other of the five solutions mentioned above.

Following Hardy and Titchmarsh I will say that a function is R_ν if it is self-reciprocal for J_ν transforms, and it is $-R_\nu$ if it is skew-reciprocal for J_ν transforms; also, for $R_{\frac{1}{2}}$ and $R_{-\frac{1}{2}}$ I will write R_s and R_c respectively.

2. Before proceeding to the examples, it seems desirable to state, very briefly, the five solutions of (1.1) mentioned above. All these formulæ are subject to appropriate conditions in each case.

(i) If $f(x)$ is $\pm R_\nu$,

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\,\psi(s)\,x^{-s}\,ds, \tag{A}$$

where

$$\psi(s)=\pm\psi(1-s), \qquad \begin{matrix}(2.1)\\(2.2)\end{matrix}$$

Here, by Mellin's Inversion Formula,

$$2^{\frac{1}{2}s}\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\,\psi(s)=\int_0^\infty x^{s-1}\,f(x)\,dx.^* \tag{2.3}$$

(ii) If $f(x)$ is $\pm R_\nu$,

$$f(x)=\frac{x^{\frac{1}{2}-\nu}}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\frac{1}{2}x^2 s}\,s^{-\frac{1}{2}(\nu+1)}\,\mu(s)\,ds, \tag{B}$$

where

$$\mu(s)=\pm\mu\left(\frac{1}{s}\right). \qquad \begin{matrix}(2.4)\\(2.5)\end{matrix}$$

By Mellin's Inversion Formula

$$\mu(s)=s^{\frac{1}{2}(\nu+1)}\int_0^\infty x^{\frac{1}{2}+\nu}\,e^{-\frac{1}{2}sx^2}\,f(x)\,dx.^\dagger \tag{2.6}$$

(iii) The function

$$f(x)=x^{\frac{1}{2}+\nu}\int_0^\infty \omega^{\nu}\,e^{-\frac{1}{2}x^2\omega^2}\,F(\omega)\,d\omega, \tag{B$'$}$$

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* Hardy and Titchmarsh (6), § 1.4.

† Hardy and Titchmarsh (6), § 1.5.

where $$F(\omega)=F\left(\frac{1}{\omega}\right), \qquad (2.7)$$

is R_ν.[*]

(iv) If $f(x)$ is R_ν,

$$f(x)=\tfrac{1}{2}x^{\frac{3}{2}}\int_0^\infty \sqrt{t}\, J_{\frac{1}{2}\nu}\left(\tfrac{1}{2}x^2t\right)\psi(t)\,dt, \qquad \text{(C)}$$

where $$\psi(t)=\psi\left(\frac{1}{t}\right)^{\dagger}.$$

(v) If $f(x)$ is R_ν,

$$f(x)=\tfrac{1}{2}\sqrt{x}\int_0^\infty \frac{1}{\sqrt{t}}\, J_{\frac{1}{2}\nu}\left(\tfrac{1}{2}x^2t\right)\Psi(t)\,dt, \qquad \text{(D)}$$

where $$\Psi(t)=\Psi\left(\frac{1}{t}\right)^{\dagger}.$$

3. *Examples:*

(i) Putting $F(\omega)=\omega^{a-\frac{1}{2}}+\omega^{\frac{1}{2}-a}$ $(o<a<1)$,

in (B′) we get

$$f(x)=2^{\frac{1}{2}\nu-\frac{3}{4}}\left\{2^{\frac{1}{2}a}\Gamma\left(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}a\right)x^{-a}+2^{\frac{1}{2}-\frac{1}{2}a}\Gamma\left(\tfrac{3}{4}+\tfrac{1}{2}\nu-\tfrac{1}{2}a\right)x^{a-1}\right\}.^{\ddagger}$$

In particular, when $a=\frac{1}{2}$,

$$F(\omega)=2, \qquad f(x)=2^{\frac{1}{2}(\nu+1)}\,\Gamma\left(\tfrac{1}{2}\nu+\tfrac{1}{2}\right).\,\frac{1}{\sqrt{x}}.^{\|}$$

(ii) Putting in (C)

$$\psi(t)=\frac{2^\nu}{\sqrt{\pi}}\Gamma\left(\tfrac{1}{2}+\tfrac{1}{2}\nu\right)\left(t+\frac{1}{t}\right)^{-\frac{1}{2}\nu-\frac{1}{2}}$$

and using a formula given by Watson,[⁑] we get

$$f(x)=x^{\frac{1}{2}+\nu}\,e^{-\frac{1}{2}x^2}.^{\S}$$

We get the same result by putting in (D)

$$\Psi(t)=\frac{2^{\nu+2}}{\sqrt{\pi}}\Gamma\left(\tfrac{3}{2}+\tfrac{1}{2}\nu\right)\left(t+\frac{1}{t}\right)^{-\frac{1}{2}\nu-\frac{3}{2}}.$$

---

[*] Bailey (1). *See also* Bailey (2) and Mehrotra (9) § 5 (iii).

[†] *See* Mehrotra (10), §§ 8—9.

[‡] *See* Hardy and Titchmarsh (6), § 3 (10).

[||] *See* Weber (13) 230.

[⁑] Watson (11), § 13.6 (2).

[§] *See* Weber (13) 228.

(iii) Let $f(x)=x^{\frac{1}{2}+\nu}\cos\left(\frac{1}{2}x^2-\frac{\nu+1}{4}\pi\right)$. *

Calculating $\mu(s)$ from (2.6) we get

$$\mu(s)=2^{\nu-1}\,\Gamma(\nu+1)\left\{\left(e^{\frac{\pi i}{4}}\sqrt{s}+\frac{1}{e^{\frac{\pi i}{4}}\sqrt{s}}\right)^{-\nu-1}+\left(\frac{e^{\frac{\pi i}{4}}}{\sqrt{s}}+\frac{\sqrt{s}}{e^{\frac{\pi i}{4}}}\right)^{-\nu-1}\right\},$$

which satisfies (2.4).

If $f(x)=x^{\frac{1}{2}+\nu}\sin\left(\frac{1}{2}x^2-\frac{\nu+1}{4}\pi\right)$, *

which is $-R_\nu$,

$$\mu(s)=2^{\nu-1}\,\Gamma(\nu+1)\left\{\left(e^{\frac{\pi i}{4}}\sqrt{s}+\frac{1}{e^{\frac{\pi i}{4}}\sqrt{s}}\right)^{-\nu-1}-\left(\frac{e^{\frac{\pi i}{4}}}{\sqrt{s}}+\frac{\sqrt{s}}{e^{\frac{\pi i}{4}}}\right)^{-\nu-1}\right\},$$

which satisfies (2.5).

(iv) If, in (A), we take $\psi(s)=P(\frac{1}{2}-s)$ where $P(u)$ is an even polynomial, or an even integral function of order less than 1, we find that

$$f(x)=2\sum_{o}^{\infty}\frac{\left(-\frac{1}{2}x^2\right)^n}{|\underline{n}}P(2n+\tfrac{1}{2})$$

is R_c. If $P(u)$ is a polynomial,

$$f(x)=e^{-\frac{1}{2}x^2}Q(x),$$

where $Q(u)$ is a polynomial. The simplest example, after $P(u)=1$, is $P(u)=u^2$, when

$$f(x)=2\,e^{-\frac{1}{2}x^2}\left(x^4-3x^2+\tfrac{1}{4}\right).\ †$$

---

* *See* Bailey (1), § (4.1) and (4.2),

† Hardy and Titchmarsh (6), § 3 (2).

(v) If $f(x) = \sqrt{x}\, J_{\frac{1}{2}\nu}(\tfrac{1}{2}x^2)$,

using a formula given by Watson*, we get, from (2.3),

$$\psi(s) = \frac{2^{-\frac{1}{2}\nu-\frac{1}{4}}\sqrt{\pi}}{\Gamma(\frac{5}{8}+\frac{1}{4}\nu+\frac{1}{4}s)\,\Gamma(\frac{7}{8}+\frac{1}{4}\nu-\frac{1}{4}s)},$$

where $0 < R(\frac{1}{4}+\frac{1}{2}\nu+\frac{1}{2}s) < R(\frac{1}{2}\nu)+\frac{3}{2}$. Obviously, $\psi(s)$ satisfies (2.1).†

(vi) If $f(x) = x^{\frac{5}{2}}\, J_{\frac{1}{2}\nu}(\tfrac{1}{2}x^2)$,

using the same formula, we get, from (2.3),

$$\psi(s) = \frac{2^{\frac{5}{4}-\frac{1}{2}\nu}\sqrt{\pi}}{\Gamma(\frac{1}{8}+\frac{1}{4}\nu+\frac{1}{4}s)\,\Gamma(\frac{3}{8}+\frac{1}{4}\nu-\frac{1}{4}s)},$$

where $0 < R(\frac{5}{4}+\frac{1}{2}\nu+\frac{1}{2}s) < R(\frac{1}{2}\nu)+\frac{3}{2}$. Here too $\psi(s)$ satisfies (2.1).‡

(vii) Let
$$f(x) = x^{\frac{1}{2}-\nu}\left(x^2-b^2\right)^{\frac{1}{4}(\nu-1)} J_{\frac{1}{2}(\nu-1)}\left(b\sqrt{x^2-b^2}\right) \qquad (x>b>0),$$
$$= 0 \qquad (0<x<b),$$

Using a formula given by Watson,|| we get, from (2.6),

$$\mu(s) = b^{\frac{1}{2}(\nu-1)}\, e^{-\frac{1}{2}b^2\left(s+\frac{1}{s}\right)},$$

which satisfies (2.4).§

(viii) Let
$$f(x) = x^{\frac{1}{2}+\nu}(x^2+a^2)^{-\frac{1}{4}(\nu+1)} K_{\frac{1}{2}(\nu+1)}\left(a\sqrt{x^2+a^2}\right) \qquad (a>0).$$

That this function is R_ν, results by elementary substitutions, from a formula given by Watson.

From (2.6), we get

$$2^{\frac{1}{2}s}\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\,\psi(s) = \int_0^\infty x^{s+\nu-\frac{1}{2}} \frac{K_{\frac{1}{2}(\nu+1)}(a\sqrt{x^2+a^2})}{(x^2+a^2)^{\frac{1}{4}(\nu+1)}}\,dx.$$

Evaluating this integral by another formula given by Watson,⁂ we get

$$\psi(s) = 2^{-\frac{5}{4}}\left(\frac{2}{a}\right)^{\frac{1}{2}(\nu+1)} K_{\frac{1}{2}(s-\frac{1}{2})}(a^2),$$

---

* Watson (11), § 13.24 (1).

† Hardy and Titchmarsh (6), § 3 (8).

‡ Titchmarsh—unpublished manuscript.

|| Watson (11), § 13.3 (4).

§ Hardy and Titchmarsh (6), § 3 (7).

‡ Watson (11). § 13.47 (2).

⁂ Watson (11), § 13.47 (6).

where $0 \leqslant -\frac{1}{2}-\nu < R(s)$. As $K_\mu(x)$ is an even function of μ, it follows that $\psi(s)$ satisfies (2.1).

(ix) The function $f_1(x) = \frac{x}{\sqrt{2\pi}} - \left[\frac{x}{\sqrt{2\pi}}\right]$, $f_1(x) = \frac{1}{2}$, according as x is not or is a multiple of $\sqrt{2\pi}$, is R_c. In this case the integral in (2.3) has to be interpreted as a Stieltjes Integral. We have

$$2^{\frac{1}{2}s}\,\Gamma(\tfrac{1}{2}s)\,\psi(s) = \lim_{N\to\infty}\int_0^{(N+\frac{1}{2})a} x^{s-1}\,df_1(x) \qquad (a=\sqrt{2\pi})$$

$$= \lim_{N\to\infty}\left[\frac{1}{a}\int_a^{(N+\frac{1}{2})a} x^{s-1}\,dx - \sum_1^N (na)^{s-1}\right]$$

$$= \lim_{N\to\infty} a^{s-1}\left[\frac{(N+\frac{1}{2})^s}{s} - \sum_1^N n^{s-1}\right] = -(2\pi)^{\frac{1}{2}s-\frac{1}{2}}\zeta(1-s).$$

Hence $\psi(s) = -\dfrac{1}{\sqrt{2\pi}}\,\dfrac{\pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)}\,\zeta(1-s)$

satisfies (2.1).*

(x) Let $r(n)$ be the number of representations of n as a sum of two squares and let

$$\overline{P}(x) = \sum_{0\leqslant n\leqslant x}{}' r(n) - \pi x, \qquad f(x) = x^{-\frac{3}{2}}\left[\overline{P}\left(\frac{x^2}{2\pi}\right) - 1\right],$$

the dash implying the insertion of a factor $\frac{1}{2}$ in the last term of the sum when x is an integer. Then $f(x)$ is R_2. From (2.3),

$$2^{\frac{1}{2}s}\,\Gamma(\tfrac{5}{4}+\tfrac{1}{2}s)\,\psi(s) = \tfrac{1}{2}(2\pi)^{\frac{1}{2}s-\frac{3}{4}}\,\frac{Z(\frac{3}{4}-\frac{1}{2}s)}{\frac{3}{4}-\frac{1}{2}s},$$

where $Z(s)$ is the function defined for $\sigma > 1$ by

$$Z(s) = \sum \frac{r(n)}{n^s} = \zeta(s)\,L(s)$$

$$= \zeta(s)\,\left(1^{-s} - 3^{-s} + 5^{-s} - 7^{-s} + \ldots\ldots\right).$$

Hence $\psi(s) = 2^{-\frac{7}{4}}\,\pi^{\frac{1}{2}s-\frac{3}{4}}\,\dfrac{\zeta(\frac{3}{4}-\frac{1}{2}s)\,L(\frac{3}{4}-\frac{1}{2}s)}{(\frac{3}{4}-\frac{1}{2}s)\,\Gamma(\frac{5}{4}+\frac{1}{2}s)}$,

which satisfies (2.1).†

(xi) Let $f(x) = x^{\frac{1}{2}+\nu}\,e^{-\frac{1}{2}x^2}\displaystyle\sum_{r=0}^{n}\frac{(-1)^{n-r}\,x^{2r}}{|\underline{r}\;|\underline{n-r}\;\Gamma(r+\nu+1)}$.‡

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* Hardy and Titchmarsh (6), § 5.4.

† *See* Hardy (5) and Hardy and Titchmarsh (6), § 3 (9).

‡ This is a trivial transformation of the function given by Wilson (14).

This function is $\pm R_\nu$ according as n is an even or odd integer. From the formal point of view, at least, we get, from (2.6),

$$\mu(s)=\sum_{r=o}^{n}\frac{(-1)^{n-r}\,s^{\frac{1}{2}\nu+\frac{1}{2}}\,2^{r+\nu}}{\underline{|r}\,\underline{|n-r}\,(s+1)^{r+\nu+1}}$$

$$=\frac{(-1)^n\,2^\nu}{\underline{|n}\left(\sqrt{s}+\frac{1}{\sqrt{s}}\right)^{\nu+1}}\sum_{r=o}^{n}(-1)^n\,{}^nC_r\left(\frac{2}{s+1}\right)^r$$

$$=\frac{(-1)^n\,2^\nu}{\underline{|n}\left(\sqrt{s}+\frac{1}{\sqrt{s}}\right)^{\nu+1}}\left(1-\frac{2}{s+1}\right)^n=\frac{(-1)^n\,2^\nu}{\underline{|n}\left(\sqrt{s}+\frac{1}{\sqrt{s}}\right)^{\nu+1}}\left(\frac{s-1}{s+1}\right)^n.$$

Here, $\mu(s)$ clearly satisfies the equation

$$\mu(s)=(-1)^n\,\mu\left(\frac{1}{s}\right)$$

(xii) Let $f(x)=2\sum_{n=1}^{\infty}\left(nx\sqrt{\frac{\pi}{2}}\right)^{\frac{1}{2}\nu+\frac{1}{4}}K_{\frac{1}{2}\nu+\frac{1}{4}}(nx\sqrt{2\pi})-\frac{\Gamma(\frac{1}{2}\nu+\frac{3}{4})}{x\sqrt{2}}$.*

Here, from (2.3),

$$2^{\frac{1}{2}s}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\psi(s)=2\sum_{n=1}^{\infty}\int_0^\infty x^{s-1}\left(nx\sqrt{\frac{\pi}{2}}\right)^{\frac{1}{2}\nu+\frac{1}{4}}$$

$$K_{\frac{1}{2}\nu+\frac{1}{4}}(nx\sqrt{2\pi})\,dx-\frac{\Gamma(\frac{1}{2}\nu+\frac{3}{4})}{\sqrt{2}}\int_0^\infty x^{s-2}\,dx.$$

Evaluating the first term on the right-hand side by a formula given by Watson,† we get

$$\pi^{-\frac{1}{2}s}\,2^{\frac{1}{2}s-1}\,\Gamma(\tfrac{1}{2}s)\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\,\zeta(s).$$

The integral in the second term is not convergent, but by an appropriate adaptation of Riesz's logarithmic means,‡ it becomes

$$-\frac{\Gamma(\frac{1}{2}\nu+\frac{3}{4})}{\sqrt{2}}\lim_{\lambda\to\infty}\frac{1}{\log\lambda}\int_1^\lambda\frac{dw}{w}\int_{\frac{1}{w}}^{w}x^{s-2}\,dx,$$

which tends to 0 unless $s=1$, when it tends to infinity.

Hence $\psi(s)=\frac{\Gamma(\frac{1}{2}s)}{2\pi^{\frac{1}{2}s}}\,\zeta(s),$

which satisfies (2.1).

---

* This is a trivial deviation from the result given by Watson (12).

† Watson (11), § 13.21 (8).

‡ *See* Hardy and Titchmarsh (6), § 3 (10).

For the particular case $\nu=\frac{1}{2}$, this function becomes a constant multiple of the familiar R_s function

$$\frac{1}{e^{x\sqrt{2\pi}}-1}-\frac{1}{x\sqrt{2\pi}}.^{*} \tag{3.1}$$

(xiii) Let $f(x)=\sum_{n=1}^{\infty}(-1)^{n-1}\Big[(2n-1)x\Big]^{\frac{1}{2}\nu+\frac{3}{4}}$

$$K_{\frac{1}{2}\nu-\frac{1}{4}}\Big[(2n-1)x\sqrt{\frac{\pi}{2}}\Big].^{\dagger}$$

Here, $\psi(s)=2^{\frac{3}{4}\nu-\frac{7}{8}}.\,\pi^{-\frac{1}{4}\nu-\frac{3}{8}}.\left(\frac{4}{\pi}\right)^{\frac{1}{2}s}\Gamma(\tfrac{1}{2}+\tfrac{1}{2}s)\,L(s)$

where $L(s)=\frac{1}{1^s}-\frac{1}{3^s}+\frac{1}{5^s}-\frac{1}{7^s}+\ldots\ldots$

That $\psi(s)$ satisfies (2.1), follows from the functional equation of $L(s)$.

For the particular case $\nu=-\frac{1}{2}$, the function becomes

$$2^{-\frac{5}{4}}\pi^{\frac{1}{4}}\frac{1}{\cosh\left(x\sqrt{\frac{\pi}{2}}\right)}.^{\ddagger} \tag{3.2}$$

(xiv) Let $f(x)=$

$$\sum_{r=0}^{\infty}(-1)^r\Big\{(4r+1)x\Big\}^{\frac{1}{2}\nu+\frac{3}{4}}K_{\frac{1}{2}\nu-\frac{1}{4}}\Big\{(4r+1)\frac{x\sqrt{\pi}}{2}\Big\}$$

$$+\sum_{r=0}^{\infty}(-1)^r\Big\{(4r+3)x\Big\}^{\frac{1}{2}\nu+\frac{3}{4}}K_{\frac{1}{2}\nu-\frac{1}{4}}\Big\{(4r+3)\frac{x\sqrt{\pi}}{2}\Big\}.$$

Here. $\psi(s)=\frac{2^{\nu-\frac{1}{2}}}{\pi^{\frac{1}{4}\nu+\frac{3}{8}}}.\frac{2^{\frac{3}{2}s}}{\pi^{\frac{1}{2}s}}.\,\Gamma(\tfrac{1}{2}+\tfrac{1}{2}s)\,\mathrm{L}_1(s)$,

where $\mathrm{L}_1(s)=\sum_{r=0}^{\infty}(-1)^r\Big[\frac{1}{(4r+1)^s}+\frac{1}{(4r+3)^s}\Big]$

$$=\frac{1}{1^s}+\frac{1}{3^s}-\frac{1}{5^s}-\frac{1}{7^s}+\frac{1}{9^s}+\frac{1}{11^s}-\ldots\ldots$$

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[*] The value of the integral in (2.3) when $f(x)$ denotes the function (3.1) has been given by Gram. (4).

[†] I remark that examples (*xiii*)—(*xvii*) have occured to me only recently. In what follows I have proved, only from the formal point of view, that they are R_ν.

[‡] Hutchinson (7)

Hence we are required to prove that $\psi(s)$ satisfies (2.1); in other words, that

$$2^{3s-\frac{3}{2}}\,\Gamma(\tfrac{1}{2}+\tfrac{1}{2}s)\ \mathrm{L}_1(s)=\pi^{s-\frac{1}{2}}\,\Gamma(1-\tfrac{1}{2}s)\ \mathrm{L}_1(1-s). \tag{3.3}$$

Hutchinson* has proved recently that if

$$Z(a, b, s)=Z(s)=\sum_{r=o}^{\infty}\frac{1}{(ar+b)^{s}},$$

then

$$Z_b+Z_{a-b}=\frac{2\pi^{s-\frac{1}{2}}}{a^{s}}\cdot\frac{\Gamma(\frac{1}{2}-\frac{1}{2}s)}{\Gamma(\frac{1}{2}s)}\sum_{n=1}^{\infty}\frac{\cos\dfrac{2bn\pi}{a}}{n^{1-s}}, \tag{3.4}$$

$$Z_b-Z_{a-b}=\frac{2\pi^{s-\frac{1}{2}}}{a^{s}}\cdot\ \frac{\Gamma(1-\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}s)}\sum_{n=1}^{\infty}\frac{\sin\dfrac{2bn\pi}{a}}{n^{1-s}}. \tag{3.5}$$

Now, $\mathrm{L}_1(s)$ may be thrown into the form

$$\left[\sum_{r=o}^{\infty}\frac{1}{(8r+1)^{s}}-\sum_{r=o}^{\infty}\frac{1}{(8r+7)^{s}}\right]+\left[\sum_{r=o}^{\infty}\frac{1}{(8r+3)^{s}}-\sum_{r=o}^{\infty}\frac{1}{(8r+5)^{s}}\right].$$

Putting in (3.5) $b=1$, $a=8$ and $b=3$, $a=8$, successively, we get

$$\begin{aligned}\mathrm{L}_1(s)&=\frac{2\pi^{s-\frac{1}{2}}}{8^{s}}\ \frac{\Gamma(1-\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}s)}\left[\sum_{n=1}^{\infty}\frac{\sin\dfrac{n\pi}{4}}{n^{1-s}}+\sum_{n=1}^{\infty}\frac{\sin\dfrac{3n\pi}{4}}{n^{1-s}}\right]\\&=\frac{\pi^{s-\frac{1}{2}}}{2^{3s-\frac{3}{2}}}\cdot\frac{\Gamma(1-\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}s)}\ \mathrm{L}_1(1-s),\end{aligned}$$

after a little algebra. This equation is equivalent to (3.3).

For the particular case $\nu=-\frac{1}{2}$, the function becomes

$$\pi^{\frac{1}{4}}\ \frac{\cosh\left(\dfrac{x\sqrt{\pi}}{2}\right)^{*}}{\cosh\ (x\sqrt{\pi})}. \tag{3.6}$$

(xv) Let $f(x)=\displaystyle\sum_{r=o}^{\infty}\Big[(3r+1)x\Big]^{\frac{1}{2}\nu+\frac{3}{4}}K_{\frac{1}{2}\nu-\frac{1}{4}}\left[(3r+1)x\sqrt{\frac{2\pi}{3}}\right]$

$$-\sum_{r=o}^{\infty}\Big[(3r+2)x\Big]^{\frac{1}{2}\nu+\frac{3}{4}}K_{\frac{1}{2}\nu-\frac{1}{4}}\left[(3r+2)x\sqrt{\frac{2\pi}{3}}\right].$$

From (2.3),

$$\psi(s)=2^{\frac{1}{4}\nu-\frac{13}{8}}\left(\frac{3}{\pi}\right)^{\frac{1}{4}\nu+\frac{3}{8}+\frac{1}{2}s}\Gamma(\tfrac{1}{2}+\tfrac{1}{2}s)\ \mathrm{L}_2(s),$$

---

* Hutchinson (7).

where $$L_2(s)=\sum_{r=0}^{\infty}\left[\frac{1}{(3r+1)^s}-\frac{1}{(3r+2)^s}\right]$$

$$=\frac{1}{1^s}-\frac{1}{2^s}+\frac{1}{4^s}-\frac{1}{5^s}+\frac{1}{7^s}-\frac{1}{8^s}+\ldots\ldots$$

We are now to prove that $\psi(s)$ satisfies (2.1), or that

$$3^{s-\frac{1}{2}}\,\Gamma(\tfrac{1}{2}+\tfrac{1}{2}s)\;L_2(s)=\pi^{s-\frac{1}{2}}\,\Gamma(1-\tfrac{1}{2}s)\;L_2(1-s). \qquad (3.7)$$

Putting $b=1$, $a=3$ in (3.5) we get

$$L_2(s)=\frac{2\pi^{s-\frac{1}{2}}}{3^s}\,\frac{\Gamma(1-\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}s)}\sum_{n=1}^{\infty}\frac{\sin\frac{2n\pi}{3}}{n^{1-s}}$$

$$=\frac{\pi^{s-\frac{1}{2}}\,\Gamma(1-\frac{1}{2}s)}{3^{s-\frac{1}{2}}\,\Gamma(\frac{1}{2}+\frac{1}{2}s)}\;L_2(1-s),$$

which is equivalent to (3.7).*

For the particular case $\nu=-\frac{1}{2}$ the function becomes

$$\left(\frac{3\pi}{8}\right)^{\frac{1}{4}}\frac{\sinh\left(x\sqrt{\frac{\pi}{6}}\right)}{\sinh\left(x\sqrt{\frac{3\pi}{2}}\right)}=\frac{\left(\frac{3\pi}{8}\right)^{\frac{1}{4}}}{1+2\cosh\left(x\sqrt{\frac{2\pi}{3}}\right)}. \qquad (3.8)$$

(xvi) Let $$f(x)=\sum_{r=0}^{\infty}(-1)^r\Big[(4r+1)x\Big]^{\frac{1}{2}\nu+\frac{1}{4}}K_{\frac{1}{2}\nu+\frac{1}{4}}\left[(4r+1)\frac{x\sqrt{\pi}}{2}\right]$$

$$-\sum_{r=0}^{\infty}(-1)^r\Big[(4r+3)x\Big]^{\frac{1}{2}\nu+\frac{1}{4}}K_{\frac{1}{2}\nu+\frac{1}{4}}\left[(4r+3)\frac{x\sqrt{\pi}}{2}\right],$$

From (2.3),

$$\psi(s)=\frac{2^{\nu-\frac{3}{2}}}{\pi^{\frac{1}{4}\nu+\frac{1}{8}}},\left(\frac{8}{\pi}\right)^{\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\;L_3(s),$$

where $$L_3(s)=\sum_{r=0}^{\infty}(-1)^r\left[\frac{1}{(4r+1)^s}-\frac{1}{(4r+3)^s}\right]$$

$$=\frac{1}{1^s}-\frac{1}{3^s}-\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{9^s}-\ldots.$$

If $\psi(s)$ is to satisfy (2.1), we must have

$$2^{3s-\frac{3}{2}}\,\Gamma(\tfrac{1}{2}s)\;L_3(s)=\pi^{s-\frac{1}{2}}\,\Gamma(\tfrac{1}{2}-\tfrac{1}{2}s)\;L_3(1-s). \qquad (3.9)$$

Now, $L_3(s)$ may be thrown into the form

$$\left[\sum_{r=0}^{\infty}\frac{1}{(8r+1)^s}+\sum_{r=0}^{\infty}\frac{1}{(8r+7)^s}\right]-\left[\sum_{r=0}^{\infty}\frac{1}{(8r+3)^s}+\sum_{r=0}^{\infty}\frac{1}{(8r+5)^s}\right].$$

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* Equations (3.3) and (3.7) have also been established by Malmsten (8).

Putting in (3.4) $b=1, a=8$ and $b=3, a=8$ successively, we get

$$L_3(s)=\frac{2\pi^{s-\frac{1}{2}}}{8^s}\,\frac{\Gamma(\frac{1}{2}-\frac{1}{2}s)}{\Gamma(\frac{1}{2}s)}\left[\sum_{n=1}^{\infty}\frac{\cos\frac{n\pi}{4}}{n^{1-s}}-\sum_{n=1}^{\infty}\frac{\cos\frac{3n\pi}{4}}{n^{1-s}}\right]$$

$$=\left(\frac{\pi}{8}\right)^{s-\frac{1}{2}}\frac{\Gamma(\frac{1}{2}-\frac{1}{2}s)}{\Gamma(\frac{1}{2}s)}L_3(1-s),$$

which is equivalent to (3.9).

For the particular case $\nu=\frac{1}{2}$, the function becomes

$$\pi^{\frac{1}{4}}\,\frac{\sinh\left(\frac{x\sqrt{\pi}}{2}\right)}{\cosh\,(x\sqrt{\pi})}. \tag{3.10}$$

(xvii) Let $f(x)=\sum_{r=0}^{\infty}(-1)^r\Big[(6r+1)x\Big]^{\frac{1}{2}\nu+\frac{1}{4}}K_{\frac{1}{2}\nu+\frac{1}{4}}\Big[(6r+1)x\sqrt{\frac{\pi}{6}}\Big]$

$$-\sum_{r=0}^{\infty}(-1)^r\Big[(6r+5)x\Big]^{\frac{1}{2}\nu+\frac{1}{4}}K_{\frac{1}{2}\nu+\frac{1}{4}}\Big[(6r+5)x\sqrt{\frac{\pi}{6}}\Big].$$

From (2.3),

$$\psi(s)=2^{\frac{3}{4}\nu-\frac{13}{8}}\left(\frac{3}{\pi}\right)^{\frac{1}{4}\nu+\frac{1}{8}}\cdot\left(\frac{12}{\pi}\right)^{\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\,L_4(s),$$

where $L_4(s)=\sum_{r=0}^{\infty}(-1)^r\left[\frac{1}{(6r+1)^s}-\frac{1}{(6r+5)^s}\right]$

$$=\frac{1}{1^s}-\frac{1}{5^s}-\frac{1}{7^s}+\frac{1}{11^s}+\frac{1}{13^s}-\frac{1}{17^s}-\frac{1}{19^s}+\dots$$

If $\psi(s)$ is to satisfy (2.1) we must have

$$12^{s-\frac{1}{2}}\Gamma(\tfrac{1}{2}s)\;L_4(s)=\pi^{s-\frac{1}{2}}\Gamma(\tfrac{1}{2}-\tfrac{1}{2}s)\;L_4(1-s). \tag{3.11}$$

Now, $L_4(s)$ may be thrown into the form

$$\left[\sum_{r=0}^{\infty}\frac{1}{(12r+1)^s}+\sum_{r=0}^{\infty}\frac{1}{(12r+11)^s}\right]-\left[\sum_{r=0}^{\infty}\frac{1}{(12r+5)^s}+\sum_{r=0}^{\infty}\frac{1}{(12r+7)^s}\right]$$

Putting in (3.4) $b=1, a=12$ and $b=5, a=12$ successively, we get

$$L_4(s)=\frac{2\pi^{s-\frac{1}{2}}}{12^s}\,\frac{\Gamma(\frac{1}{2}-\frac{1}{2}s)}{\Gamma(\frac{1}{2}s)}\left[\sum_{n=1}^{\infty}\frac{\cos\frac{n\pi}{6}}{n^{1-s}}-\sum_{n=1}^{\infty}\frac{\cos\frac{5n\pi}{6}}{n^{1-s}}\right]$$

$$=\left(\frac{\pi}{12}\right)^{s-\frac{1}{2}}\frac{\Gamma(\frac{1}{2}-\frac{1}{2}s)}{\Gamma(\frac{1}{2}s)}\;L_4(1-s),$$

which is equivalent to (3.11).

For $\nu = \frac{1}{2}$, the function becomes

$$\left(\frac{3\pi}{2}\right)^{\frac{1}{4}} \frac{\sinh\left(x\sqrt{\frac{2\pi}{3}}\right)}{\cosh\left(x\sqrt{\frac{3\pi}{2}}\right)} = \left(\frac{3\pi}{2}\right)^{\frac{1}{4}} \frac{2\sinh\left(x\sqrt{\frac{\pi}{6}}\right)^{*}}{2\cosh\left(x\sqrt{\frac{2\pi}{3}}\right)-1}. \qquad (3.12)$$

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* That the functions (3.2), (3.6), (3.8) are R_c and the functions (3.10) and (3.12) are R_s, follows independently from formulae given by Glaisher (3).

"ON THE PRODUCT OF PARABOLIC CYLINDER FUNCTIONS"

BY

S. C. DHAR, D.SC.

§ 1. *Introduction.*

In a recent paper I have, by a method of transformation of integrals, obtained expressions for the product of two parabolic cylinder functions with different arguments in the form

$$D_n(X)D_m(x)=\frac{1}{(\sqrt{2})^{m+n}}\sum_{r=0}^{n+m}B_{n+m-r}D_{n+m-r}\left(\frac{X+x}{\sqrt{2}}\right)D_r\left(\frac{X-x}{\sqrt{2}}\right)$$

where B_{n+m-r} are certain co-efficients and m and n positive integers. In fact it has been proved by me that this is but a particular form of a more general expression given by

$$D_n(X)\ D_m(x)=$$

$$\sum_{r=0}^{n+m} A_{n+m-r}D_{n+m-r}\ (X\cos\alpha+x\sin\alpha)\ D_r(X\sin\alpha-x\cos\alpha),$$

where

$$A_{n+m-r}=\sin^{m+r}\alpha\cos^{n-r}\alpha\ \frac{n(n-1)\ldots\ldots(n-r+1)}{r\,!}$$

$$\times F\left(-r,-m\,;\,n-r+1\,;\,-\frac{1}{\tan^2\alpha}\right)$$

In this paper an attempt is made to study the case when m and n are not integers.

§ 2. It was proved by Watson[*] that the parabolic cylinder functions $D_{-n-1}(-iz)$ is given by the integral

$$D_{-n-1}(-iz)=\frac{2^{n+1}}{\Gamma(n+1)}\ e^{\frac{1}{4}z^2}\int_{-\infty}^{\infty}e^{-2u^2+2izu}\ u^n\ du,\ldots\ldots(1)$$

when arg. $u=0$ and $n>-1$.

From this it is easy to show that when $n>-1$

$$D_n(z)=(-1)^{\frac{1}{2}n}\,2^{n+1}(2\pi)^{-\frac{1}{2}}\ e^{\frac{1}{4}z^2}\int_{-\infty}^{\infty}e^{-2u^2+2izu}\ u^n\ du\quad(2)$$

---

[*] Watson: Proc. Lon. Math. Soc., Vol. VIII, p. 404.

Hence when $n>-1$ and $m>-1$ and X and x are any two quantities, the product of two parabolic cylinder functions can be written as

$$D_n(X)\,D_m(x)$$
$$=(-1)^{\frac{1}{2}(n+m)}\,\frac{2^{n+m+1}}{\pi}\,e^{\frac{1}{4}(X^2+x^2)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-2u^2-2t^2+2iXu+2ixt}\,u^n\,t^m\,du\,dt$$
$$=(-1)^{\frac{1}{2}(n+m)}\,\frac{2^{n+m+1}}{\pi}\,e^{\frac{1}{4}(X^2+x^2)}\underset{R\to\infty}{Lt}\iint_{S_R}e^{-2u^2-2t^2+2iXu+2ixt}\,u^n\,t^m\,du\,dt\;\ldots.(3)$$

where S_R denote the square

$$-R\leqslant u\leqslant R$$
$$-R\leqslant t\leqslant R$$

§ 3. Now, transform the integral by the following substitution

$$\left.\begin{aligned}u\sin\alpha-t\cos\alpha&=T\\ u\cos\alpha+t\sin\alpha&=U\end{aligned}\right\}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

with the special value $\alpha=\frac{\pi}{4}$, which is unnecessary for the present discussion.

The double integral (3) integrated over the area given by

$$t=0,\ u=R\ \text{and}\ u\sin\alpha+t\cos\alpha=R\sqrt{2}\ \left(\text{when}\ \alpha=\frac{\pi}{4}\right)$$

gives

$$\int_0^R e^{-2t^2+2ixt}\,t^m\,dt\int_R^{2R-t}e^{-2u^2+2iXu}\,u^n\,du.$$

Now,

$$\int_R^{2R-t}e^{-2u^2+2iXu}\,u^n\,du=\left[\frac{1}{2iX}e^{-2u^2+2iXu}\,u^n\right]_R^{2R-t}$$
$$-\frac{1}{2iX}\int_R^{2R-t}e^{2iXu}\,\frac{d}{du}\left[e^{-2u^2}\,u^n\right]du$$
$$=O\left(e^{-2R^2}\right)+O\left(e^{-2R^2}\right)$$

and the double integral over the triangle is less in modulus than

$$O\left(e^{-2R^2}\right)\int_0^R e^{-2t^2}\,t^m\,dt=O(1).$$

Similarly the double integral over the area

$$u=0,\ t=R \text{ and } u \sin \alpha+t \cos \alpha=R\sqrt{2}$$

will give the same result.

Hence if we denote by T_R the area enclosed by

$$-R\sqrt{2}<U<R\sqrt{2}$$
$$-R\sqrt{2}<T<R\sqrt{2}$$

the integral (3) becomes

$$D_n(X)\,D_m(x)$$
$$=(-1)^{\frac{1}{2}(n+m)}\frac{2(\sqrt{2})^{n+m}}{\pi}e^{\frac{1}{4}(X^2+x^2)}\mathop{Lt}_{R\to\infty}\iint_{T_R} F d\dot{U}\,dT$$

$$\left.\begin{array}{l} =(-1)^{\frac{1}{2}(n+m)}\dfrac{2(\sqrt{2})^{n+m}}{\pi}e^{\frac{1}{4}(X^2+x^2)}\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F d\bar{U}\,dT \\ \text{where} \\ F=(U+T)^n(U-T)^m e^{-2(U^2+T^2)+\sqrt{2}i(X+x)U+\sqrt{2}i(X-x)T} \end{array}\right\}$$
$$\dots\dots\dots\dots\dots\dots(5)$$

§ 4. Take the special case when $X=x$, then the integral (5) reduces to

$$D_n(x)\,D_m(x)$$
$$=(-1)^{\frac{1}{2}(n+m)}\frac{2(\sqrt{2})^{n+m}}{\pi}e^{\frac{1}{2}x^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(U+T)^n(U-T)^m$$
$$\times e^{-2(U^2+T^2)+2\sqrt{2}ixU}\,dU\,dT\dots\dots(6)$$

Now

$$(U+T)^n(U-T)^m$$

$$\left.\begin{array}{l} =\displaystyle\sum_{r=0}^{\infty}B_{n+m-r}T^{n+m-r}U^r, \\ \text{where} \\ B_{n+m-r}=(-1)^m\dfrac{n(n-1)\dots(n-r+1)}{r!} \\ \qquad\qquad F(-r,-m;n-r+1;-1) \end{array}\right\}\dots(7)$$

Hence from (6) and (7) we get

$$D_n(x)\,D_m(x)$$
$$=(-1)^{\frac{1}{2}(n+m)}\frac{2(\sqrt{2})^{n+m}}{\pi}\sum_{r=0}^{\infty}B_{n+m-r}$$
$$\times\int_{-\infty}^{\infty}e^{-2U^2+2\sqrt{2}ixU+\frac{1}{2}x^2}U^r\,dU\int_{-\infty}^{\infty}e^{-2T^2}T^{n+m-r}\,dT$$

(term by term integration is justifiable)

$$=(-1)^{\frac{1}{2}(n+m)}\frac{(\sqrt{2})^{n+m+1}}{\sqrt{\pi}}\sum_{r=0}^{\infty}\frac{(-1)^{-\frac{1}{2}r}}{2^{r}}B_{n+m-r}\,D_{r}\,(x\sqrt{2})$$

$$\times\int_{-\infty}^{\infty}e^{-2T^{2}}\,T^{n+m-r}\,dT\,\ldots\ldots\ldots\,(8)$$

Now, the integral

$$\int_{-\infty}^{\infty}e^{-2T^{2}}\,T^{n+m-r}\,dT$$

$$=\frac{2^{r-n-m-1}\,(-1)^{-\frac{1}{2}(n+m)}}{(n+m-r+1)\,(n+m-r+3)\,..\,(n+m-1)}\sqrt{2\pi}\,D_{n+m}(0),$$

(when r is even)

$$=\frac{2^{r-n-m-1}\,(-)^{-\frac{1}{2}(n+m+1)}}{(n+m-r+1)\,(n+m-r+3)\,..\,(n+m)}\sqrt{2\pi}\,D_{n+m+1}(0)$$

(when r is odd)

Hence from (8) we get,

$$D_{n}(x)\,D_{m}(x)=D_{n+m}(0)\,\frac{1}{(\sqrt{2})^{n+m}}\sum_{0=r}^{\infty}$$

$$\frac{(-1)^{-r}}{(n+m-2r+1)\,(n+m-2r+3)\,..\,(n+m-1)}\,B_{n+m-2r}\,D_{2r}(x\sqrt{2})$$

$$+D_{n+m+1}(0)\,\frac{1}{(\sqrt{2})^{n+m}}\sum_{r=0}^{\infty}$$

$$\frac{(-1)^{-r-1}}{(n-m-2r)\,(n+m-2r+2)\,..\,(n+m)}\,B_{n+m-2r-1}\,D_{2r+1}(x\sqrt{2})$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

Now, if n and m are both positive integers and are equal, then since $D_{2n+1}(0)=0$, we get

$$D_{n}^{2}(x)=\frac{1}{2^{n}}\sum_{r=0}^{n}\frac{n!\,(2r-1)\,(2r-3)\ldots3\,.\,1}{r!\,(n-r)\,!}\,D_{2n-2r}(x\sqrt{2})\quad(10)^{*}$$

on simplification from (9).

August, 1934.

Cf. Mitra: Edin. Math. Soc., Vol. IV, p. 32.

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